TRANSVERSE IMPACT ON CANTILEVER HAVING A MASS ATTACHED AT THE FREE END

KAMPAL MISRA

PHYSICS DEPARTMENT, UNIVERSITY COLLEGE OF ENGINEERING, BURLA, ORISSA-INDIA.

(*Received February* 10, 1968)

ABSTRACT. The dynamics of vibration of a cantilever having a mass attached at its free end and struck transversely by an elastic load has been worked out employing operationa method due to Heaviside. The expressions for displacement and pressure of impact in form of respective series are given. An experiment has been reported on a 90 cm. long 1.27 cm. dia. cantilever of mild steel, using photographic method. The agreement between the theory and the experiment is excellent.

INTRODUCTION

Mason (1936), Timoshenko (1956), Hoppmann (1948, 1961), and others have given reviews of the works done on the problem of transverse impact on uniform beams. Banerjee (1966), has developed a theory of a linear beam using classical equations of beams. The present author constructs a theory on the transverse impact on cantilever having a mass attached at its free end using operational method due to Heaviside.

The cantilever having a mass attached at its free end is at rest before impact begins. The elastic load is supposed to be linear in mass and spring. Pressure exerted by the load is taken to be the change in the shearing force across the struck point. The duration of impact is the lowest positive root of t, other than zero, and is obtained by solving $P = 0$ for the struck point given.

An experiment on such a beam has been performed by the author using photographic method due to Banerjee (1966). The method is simple and convincing in as much as the motion of the beam and the striking load at every stage of impact is recorded photographically. The theory which is worked out on similar line as done by Banerjee (1966) agrees with the experiment nicely.

NOMENCLATURE

l = length of theo antilever $= a+b$

x = variable measured along length of the beam. The beam is fixed at $x = 0$ and free at $x = l$.

a = segment of the beam towards the fixed end

b = segment of the beam towards the free end

t = variable time

Y_a = displacement of the struck point. In this problem $Y_a = Y_l$

Y_1 = displacement of any point, $x < a$

Y_2 = displacement of any point, $x > a$

M = mass of the cantilever

m = mass of the hammer

M_l = mass of the attached load at the free end

E_l = Young's modulus of material of the cantilever

E_2 = elastic factor of the striking load, different from Young's modulus

k = radius of gyration about neutral axis of the cantilever

I = moment of inertia of cross section of cantilever about neutral axis

c = velocity of longitudinal waves along the beam

u = compression of the hammer

z = displacement of the hammer = $Y_a + u$

v_0 = velocity of the striking load (hammer) before impact

$J = mv_0$ (initial impulse)

$nl = \gamma$ (gamma)

$D = \dfrac{d}{dt}$ (operator)

$\psi = -2\dfrac{M_l}{M}\gamma$

The equation for the transverse vibration of the beam is given by

$$\frac{d^4y}{dx^4} + \frac{D^2}{k^2c^2}\,y = 0 \qquad .. \quad (1)$$

whence

$$y = A_1 \sinh nx + A_2 \cosh nx + A_3 \sin nx + A_4 \cos nx \qquad .. \quad (2)$$

where A_1, A_2, A_3, A_4, are coefficients

For a cantilever having a mass attached at its free end, we have

at $\quad x = 0,\ y = 0,$ and $\dfrac{dy}{dx} = 0 \qquad .. \quad (3.1)$

at $\quad x = l, \; \dfrac{d^2y}{dx^2} = 0, \;$ and $\; E_1 I \dfrac{d^3y}{dx^3} = M_l D^2 y \qquad .. \quad (3.2)$

Further at the struck point, i.e., at $x = a$, we have

$$Y_1 = Y_a = Y_2 \qquad .. \quad (4.1)$$

$$\left(\frac{dY_1}{dx}\right)_{x=a} = \left(\frac{dY_2}{dx}\right)_{x=a} \qquad .. \quad (4.2)$$

$$\left(\frac{d^2Y_1}{dx^2}\right)_{x=a} = \left(\frac{d^2Y_2}{dx^2}\right)_{x=a} \qquad .. \quad (4.3)$$

From equations (3) and (4) we get

$$Y_1 = Y_a \frac{\Delta_1(\sinh nx - \sin nx) + \Delta_2(\cosh nx - \cos nx)}{\Delta_0} \qquad .. \quad (5.1)$$

$$Y_2 = Y_a \frac{\Delta_3[\sinh n(l-x) + \sin n(l-x)] + \Delta_4[\cosh n(l-x) + \cos n(l-x) + \psi \sin n(l-x)]}{\Delta_0} \qquad ... \quad (5.2)$$

where $\Delta_1 = 2[\sinh nl \sin nb - \cosh nl \cos nb - \cosh nb \cos nl - \sinh nb \sin nl - \cosh na - \cos na] + 2\psi[\sinh nb \cos nl - \cosh nl \sin nb] \qquad .. \quad (6.1)$

$$\Delta_2 = 2[\sinh nl \cos nb + \cosh nb \sin nl + \sinh na + \sin na - \cosh nl \sin nb - \sinh nb \cos nl] + 2\psi[\sinh nl \sin nb - \sinh nb \sin nl] \qquad .. \quad (6.2)$$

$$\Delta_3 = 2[\cosh nl \cos na - \cosh na \cos nl - \sinh nl \sin na - \sinh na \sin nl \; \cosh nb + \cos nb] + 2\psi[\sinh na \cos nl - \cosh na \sin nl + \sin nb] \qquad .. (6.3$$

$$\Delta_4 = 2[\cosh nl \sin na - \sinh na \cos nl - \sinh nl \cos na + \cosh na \sin nl + \sinh nb - \sin nb] \qquad .. \quad (6.4)$$

$$\Delta_0 = 4[\cosh na \cosh nb \sin nl - \sinh nl \cos na \cos nb + \sinh nb \cos nb - \cosh nb \sin nb + \cosh na \sin na - \sinh na \cos na] + 2\psi[\cosh nl \sin na \sin nb + \sinh na \sinh nb \cos nl - \sinh nl \sin nb \cos na - \cosh na \sinh nb \sin nl + 2 \sinh nb \sin nb] \qquad (6.5)$$

The pressure exerted by the impinging load is given by

$$P = m \frac{d^2z}{dt^2} = -E_2 u \qquad (7.1)$$

The motion of the beam follows the relation

$$m \frac{d^2z}{dt^2} = E_1 I \Delta \left(\frac{d^3y}{dx^3}\right)_{x=a} = E_1 I n^3 f(D) \qquad (7.2)$$

where $\Delta\left(\frac{d^3y}{dx^3}\right)_{x=a}$ denotes the change in the value of $\frac{d^3y}{dx^3}$ in crossing the struck point, $x = a$ and

where
$$f(D) = 2\frac{u_1}{v_1} \tag{8}$$

and
$$u_1 = 1+\cosh nl \cos nl+\psi/2[\cosh nl \sin nl-\sinh nl \cos nl] \tag{9.1}$$

$$\begin{aligned} v_1 = {} & \sinh nl \cos na \cos nb+\cosh nb \sin nb+\sinh na \cos na \\ & -\cosh na \cosh nb \sin nl-\sinh nb \cos nb-\cosh na \sin na \\ & +\psi/2[\cosh na \sinh nb \sin nl-\cosh nl \sin na \sin nb \\ & -\sinh na \sinh nb \cos nl+\sinh nl \sin nb \cos na-2 \sinh nb \sin nb] \end{aligned} \tag{9.2}$$

From equations (7.1) and (7.2) we write

$$mD^2Y_a+mD^2u-E_1IY_an^3f(D) = DJ \tag{10.1}$$

and
$$mD^2Y_a+mD^2u+E_2u = DJ \tag{10.2}$$

Solving these equations we get,

$$u = -\frac{E_1I}{E_2}Y_an^3f(D) \tag{11}$$

and
$$Y_a = \frac{F(D)}{F_1(D)}v_0 \tag{12}$$

where $F(D)$ stands for D and $F_1(D)$ has the value

$$D^2-\frac{E_1I}{m}n^3\left(1+\frac{mD^2}{E_2}\right)f(D) \tag{13}$$

With the help of Heaviside's expansion theorem, we write

$$\frac{Y_a}{v_0} = \frac{F(0)}{F_1(0)}+\sum\frac{F(\alpha_s)}{\alpha_sF_1'(\alpha_s)}e^{\alpha_st} \tag{14}$$

where summation extends over all roots of $D = [\alpha_s]$ for $s = 1, 2, 3, \ldots, r$. For roots of D from $F_1(D) = 0$, we have $F_1(D) = 0$, whence

$$f(D) = -\frac{m}{M}\gamma\Big/1-\frac{E_1I}{E_2}\frac{m}{M}\frac{\gamma^4}{l^3} \tag{15}$$

Equation (15) can be solved by plotting the curve

$$\eta_1 = \frac{w}{z} \quad .. \quad (16.1)$$

and

$$\eta_2 = \frac{m}{M}\gamma / 2\left(1 - \frac{E_1 I}{E_2}\frac{m}{M}\frac{\gamma^4}{l^3}\right) \quad .. \quad (16.2)$$

The points of intersections of these curves gives the values of γ.

For hard load $E_2 \to \infty$, and $\eta_2 - \gamma$, curve is a straight line passing through the origin having an inclination with γ-axis, such that $\tan\theta = \frac{m}{M}$

Thus, $nl = \gamma_s(\gamma_s$ a pure number for $s = 1, 2, 3, \ldots r)$ .. (17.1)

$$D = [\alpha_s] = \pm iq_s \quad .. \quad (17.2)$$

Therefore,

$$q_s = \gamma_s^2\left(\frac{E_1 I}{M l^3}\right)^{\frac{1}{2}} \quad .. \quad (17.3)$$

The equation (13) can be written as,

$$F_1(D) = D^2 + \frac{2M}{m}D^2\left(1 + \frac{mD^2}{E_2}\right)\frac{u_1}{\gamma v_1}$$

Further, we have $F(0) = 0$, and $F_1(0) \neq 0$, putting $D = 0$.

Hence, with the help of Heaviside's expansion theorem (eqn. 14) and after simplification we get,

$$Y_a = 4v_0 \sum \frac{A_s}{q_s}\sin q_s t \quad .. \quad (18)$$

where for hard load,

$$\frac{1}{A_s} = 1 + \frac{M + M_L}{m} + \frac{2\gamma - \frac{M}{m}\psi}{\coth\gamma - \cot\gamma_s} \quad .. \quad (19)$$

PREESSRE EXERTED BY LOAD

The pressure of impact is given by

$$P = -mv_0\frac{F(D)}{F_2(D)} \quad .. \quad (20)$$

where
$$F_2(D) = 1 + \frac{mD^2}{E_2} + \frac{\frac{m}{M}\gamma}{f(D)} \quad .. \quad (21)$$

Using Heaviside's expansion theorem, we have

$$P = -mv_0\left[\frac{F(0)}{F_2(0)} + \sum \frac{F(\alpha_s)}{F_2'(\alpha_s)} e^{\alpha_s t}\right. \quad .. \quad (22)$$

where summation extends over all roots of $D = [\alpha_s] = \pm iq_s$ For roots of D from $F_2(D) = 0$, we have, $F_2(D) = 0$, whence

$$f(D) = -\frac{m}{M}\gamma \Big/ 1 - \frac{E_1 I}{E_2}\frac{m}{M}\frac{\gamma^4}{l^3} \quad .. \quad (23)$$

Equations (15) and (23) are identical and therefore γ_s will have same set of values as obtained previously (eqn. 16). Proceeding along similar lines as before, finally we get,

$$P = 4mv_0 \Sigma B_s q_s \sin q_s t \quad .. \quad (24)$$

where for hard load; $\frac{1}{B_s} = \frac{1}{A_s}$ and is given by equation (19),

and q_s has the same value as given by the equation (17.3).

EXPERIMENTAL

The cantilever is rigidly fixed at one end in a heavy iron pillar whose base is embedded in concrete. A solid brass sphere with a hole symmetrically drilled through it is fitted at the free end of the cantilever by means of a screw such that the centre of mass of the attached sphere coincides with the centre of section of the cantilever at its free end. The cantilever was held perfectly horizontal as was tested by means of a spirit level.

The hammer (a spherical brass bob) which is suspended from a rigid support above the beam is released from a particular distance to strike the cantilever transversely at a specified point as it swings with a particular velocity.

To record time a pointer attached to the prong of an electrically maintained tuning fork of 100 cps vibrates by the side of the cantilever. An arc lamp fitted vertically above the cantilever casts the shadow of the cantilever, the impinging hammer and the vibrating pointer on a slit which is cut on top plank of a long wooden box. The box is placed under the cantilever and parallel to it. The slit is at right angles to the length of the box. This box has within it a sliding wooden carriage which can be moved along the length of the box by means of suspended weights as in an Atwood's machine. A photographic paper pinned on the upper surface of this carriage records the shadowgraphs of the cantilever, the hammer and the pointer as the carriage moves under the slit. The shadowgraph of any section of the cantilever can be recorded by displacing the wooden box to bring the slit under the desired section with the arc lamp above it.

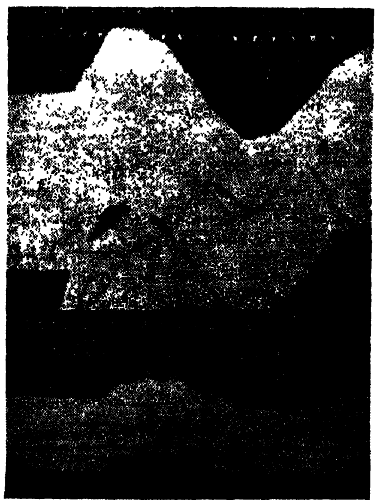
Particulars of cantilever and hammer.

Cantilever—mild steel rod, length 90 cms. dia. 1.27 cms. weight 904.5 gms.

Hammer—brass sphere. diam. 4 cms. weight = 287.8 gms.

Attached mass—brass sphere, weight 106.1 gms. diam. = 3.1 cms.

The experimental time-displacement curve is drawn to scale from the corresponding shadowgraphs. The displacement scale is obtained from the depths of shadows of the steady position of the beam towards left of each shadowgraph (for figure 1A, equivalent to 3.1 cms. and for figure 1B, equivalert to 1.27 cms.). The photographs of vibrating pointer of the tuning fork gives the time scale by juxtaposition.



Figures : 1A and 1B

Figure. 1A represents the shadowgraph of the struck point at the free end, i.e., at $x = 1$ and velocity of impact = 94.62 cms/sec. The black patches within the total contact range show separations and the white patches within this range show the contacts of the hammer and the beam. Thus the phenomenon of 'multiple contacts' is observed.

Figure 1B represents the shadowgraph of the midpoint of the cantilever when it was struck at the free end with a velocity of 78.9 cms/sec. It shows a deflection of the cantilever towards the negative direction for a short time just after the impact begins. Thereafter the beam takes the normal positive direction.

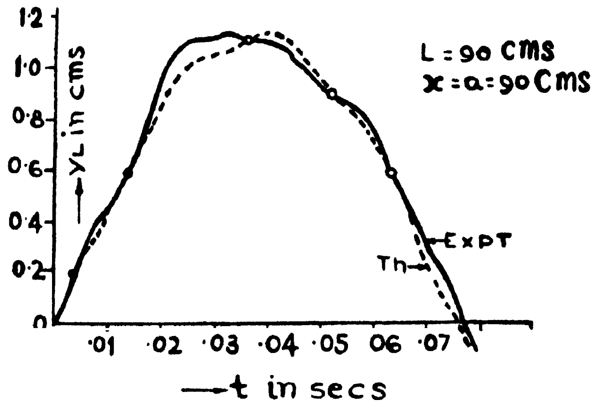


Figure : 2A

Figure 2A and 2B, are time-displacement curves drawn according to experimental curves depicted in figurs 1A and 1B respectively, and corresponding theoretical relations given by equations (18-5) From fiigure 2A it is found that the theoretical and experimental curves almost coincide upto about .011 sec. After this

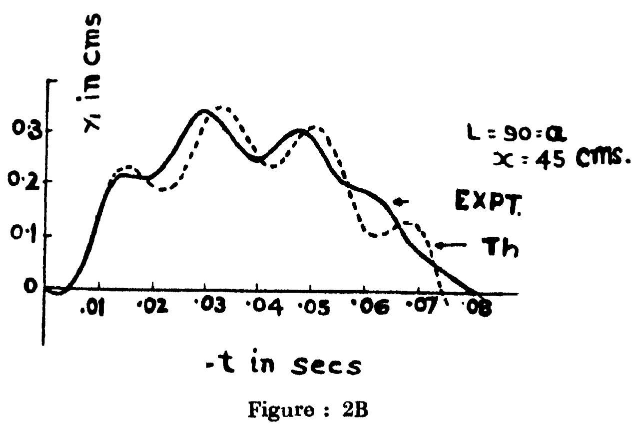


Figure : 2B

time the experimental velocity of the beam is higher which is due to sudden change in velocity suffered by the beam occuring in opposite sense to that suffered by the load at the termination of first contact and so on. Similar pheno menon was observed by Banerjee (1966) earlier in case of uniform cantilever. From fiigure 2B, it is found that the experimental and theoretical curves are strikingly similar. Both the curves coincide upto the time of first contact, thereafter, the two curves differ negligibly. The larger difference between the two curves beyond the time 0-057 sec. may be due to appreciable damping of higher modes of vibration. The effect of damping has not been considered in this analysis. The theoretical duration of impact (first contact) is .0102 sec. and its experimental value is .0105 sec which is a good agreement.

ACKNOWLEDGMENT

My best thanks are due to Dr. B. B. Banerjee of U.C.E., Burla for guidance and to Dr. M. Ghosh of City College, Calcutta for the interest he took in this work.

REFERENCES

Banerjee, B.B., 1966, *Indian J. Phys.* **40**, 198, 205.

Hoppmann, W. H., 1948, *J. Appl. Mech.* **15**, 125.

1961, *Shock and Vibration Handbook.*, Mc. Graw Hill Book Co. Inc. **1**, 9-5.

Mason, H. L., 1936, *J. Appl Mech.*, **58**, A-55.

Timoshenko, S. P., 1956, *Vibration Problems in Engineering*, 3rd. Edn. p. 413. D. Van Nostrand Co. Inc.